FURTHER IMPROVEMENTS IN WARING'S PROBLEM, II: SIXTH POWERS

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1. Introduction

In recent years there have been a series of developments in the theory of Waring's problem, following the introduction of the use of numbers with only small prime factors in Vaughan [**3**]. This has occurred through the provision of upper bounds for the number of solutions, $S_s^{(k)}(P,R)$, of the diophantine equations

$$x_1^k + \cdots + x_s^k = y_1^k + \cdots + y_s^k,$$

with $x_i, y_i \in \mathcal{A}(P,R)$ $(1 \le i \le s)$, where throughout we write

$$\mathcal{A}(P,R) = \{n \in \mathbb{Z} \cap [1,P] \ : \ p \text{ prime}, p|n \text{ implies } p \le R\}.$$

When $R = P^\eta$, with $\eta = \eta(\varepsilon, s, k)$ a sufficiently small but fixed positive number, such bounds take the form

$$S_s^{(k)}(P,R) \ll P^{\lambda_s + \varepsilon}.$$

As a consequence of further developments due to Wooley [**6**], this has led in Vaughan and Wooley [**5**] to the upper bounds

$$G(5) \le 17,\ G(6) \le 25,\ G(7) \le 33,\ G(8) \le 43,\ G(9) \le 51,$$

where, as usual, we write $G(k)$ for the smallest number s such that every sufficiently large natural number s is the sum of, at most, s kth powers of natural numbers. In the case of $k = 6$, we were able in [**5**] to prove that

$$S_{12}^{(6)}(P,R) \ll P^{18+\varepsilon}. \tag{1.1}$$

An expert in the Hardy-Littlewood method might, at first sight, expect that a further refinement would lead relatively easily to $G(6) \le 24$. However, a perusal of the methods developed in [**3**, **4**, **5**, **6**] would reveal a number of fundamental obstructions, and that the problem is one of very great delicacy.

In this paper we proceed to overcome these obstructions by

(i) the development of an efficient differencing process restricted to minor arcs,

(ii) a sequence of no fewer than five distinct pruning processes on both the major and minor arcs,

(iii) splitting certain of the exponential sums by the use of a combinatorial lemma in order to ameliorate difficulties arising from singular solutions of auxiliary congruences.

Nevertheless, and in spite of these technical difficulties, the method leads to a lower bound of the anticipated correct order of magnitude for the number of solutions of the equation

$$x_1^6 + \cdots + x_{24}^6 = n \tag{1.2}$$

in natural numbers x_i $(1 \le i \le 24)$.

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Theorem. *Let $R(n)$ denote the number of solutions of the equation* (1.2) *in natural numbers x_i* ($1 \le i \le 24$). *Then $R(n) \gg n^3$, and consequently $G(6) \le 24$.*

In section 2 we record some estimates from [**4**] and [**5**] of use in the course of our argument. We begin section 3 with a reduction process taking the minor arcs $\mathfrak{m}$ to a set of arcs $\mathfrak{n}$, and the subsequent extraction of an efficient difference. We then prune the arcs $\mathfrak{n}$ down to a set of arcs $\mathfrak{p}$ in preparation for section 4, where we set up the combinatorial machinery necessary for the extraction of a second efficient difference on the minor arcs. Thus, in section 5, after a further reduction of the arcs $\mathfrak{p}$ down to a set of arcs $\mathfrak{q}$, we are able to perform this second efficient differencing operation. We are now forced, in section 6, to prune the arcs $\mathfrak{q}$ down to a set $\mathfrak{r}$, the latter being amenable to the arguments of [**5**]. In this way, we are able to show that the contribution of the minor arcs $\mathfrak{m}$ is suitably small. Finally, we dispose of the major arcs $\mathfrak{M}$ in section 7, this entailing further (both implicit and explicit) pruning operations.

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2. Preliminaries

Our exposition will be much simplified by the adoption of certain notational conventions, which we now describe. We take $\Omega = (\delta, \nu, \gamma, \varepsilon, \eta, \tau)$ to be our basic ordered set of sufficiently small positive numbers. Let n be a sufficiently large positive integer depending at most on Ω. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on Ω. We adopt the following convention concerning the number n and elements of Ω. Whenever n, δ, ν, γ, ε, η or τ appear in a statement, either implicitly or explicitly, we assert that for each $\delta > 0$, there exist positive numbers $\nu = \nu(\delta)$, $\gamma = \gamma(\nu, \delta)$, $\varepsilon = \varepsilon(\gamma, \nu, \delta)$, $\eta = \eta(\varepsilon, \gamma, \nu, \delta)$ and $\tau = \tau(\eta, \varepsilon, \gamma, \nu, \delta)$, such that the statement holds. Note that the "value" of elements of Ω may change from statement to statement, and hence also the dependency of implicit constants on these numbers. Thus, for example, if $f \ll P^{\tau+\varepsilon\eta}$ and $g \ll P^{\nu+\gamma}$, then we shall conclude that $fg \ll P^{\nu}$ without comment. Notice that since our methods will involve only a finite number of statements, there is no danger of losing control of implicit constants through the successive changes alluded to.

Not surprisingly, the proof of our main theorem is motivated by the analysis of the iterative scheme for sixth powers which, in [**5**, section 12], led to the estimate (1.1). Before we proceed further, therefore, we record here some parameters from that analysis relevant to our subsequent deliberations. Let

$$P = n^{1/6} \quad \text{and} \quad R = P^{\eta}.$$

For $s = 1, 2, \dots$, let λ_s denote the smallest positive number for which

$$S_s(P, R) \ll P^{\lambda_s + \varepsilon},$$

and define Δ_s by $\lambda_s = 2s - k + \Delta_s$. Then by the table for $k = 6$ in [**5**, Appendix], we have $\Delta_{10} \le \Delta_{10}^*$, with $\Delta_{10}^* = 0.2030055$. Further, by the analysis of part (ii)(b) of [**5**, section 12], we have $\Delta_{11} \le \Delta_{11}^*$, where

$$\Delta_{11}^* = \Delta_{10}^*(1 - \theta_{11}) + 6\theta_{11} - 1,$$

and

$$\theta_{11} = \frac{2 - \Delta_{10}^*}{12 - \Delta_{10}^*}.$$

Consequently,

$$\Delta_{11}^* = \frac{5\Delta_{10}^*}{12 - \Delta_{10}^*}. \tag{2.1}$$

The analysis of part (iii) of [**5**, section 12] implies that $\Delta_{12} = 0$, which is tantamount to the estimate (1.1). We shall imitate that analysis, extracting two efficient differences with respective parameters θ_{12} and ϕ_{12}, which we define by

$$\theta_{12} = \frac{1 - \Delta_{11}^*}{6 - \Delta_{11}^*} = \frac{12 - 6\Delta_{10}^*}{72 - 11\Delta_{10}^*},$$

and

$$\phi_{12} = \frac{1 - \Delta_{10}^*(1 - \theta_{12})}{6 - \Delta_{10}^*}.$$

It will be apparent from the argument of [**5**, section 12] leading to [**5**, (12.55)], together with (2.1), that the parameters θ and ϕ which arise in equations (12.52) and (12.53) of [**5**] are precisely our parameters θ_{12} and ϕ_{12} respectively. For the sake of convenience, we define

$$\theta = \theta_{12} - \nu \quad \text{and} \quad \phi = \phi_{12} - \nu. \tag{2.2}$$

Then in particular, $\theta < 0.154543$ and $\phi < 0.142897$.

For $s = 1, \dots, 22$, let M_s be a real number satisfying

$$P^\theta \le M_s \le P^{\theta+\tau}, \tag{2.3}$$

and write

$$Q_s = PM_s^{-1} \quad \text{and} \quad H_s = PM_s^{-6}.$$

Consider the number $r(n;\mathbf{M}) = r(n; M_1, \dots, M_{22})$ of solutions of the equation

$$x_1^6 + x_2^6 + (p_1y_1)^6 + \dots + (p_{22}y_{22})^6 = n,$$

with the p_s $(1 \le s \le 22)$ prime numbers satisfying

$$p_s \equiv -1 \pmod 6, \quad M_s < p_s \le 2M_s, \tag{2.4}$$

and with

$$1 \le x_r \le P \quad (r = 1, 2), \qquad y_s \in \mathcal{A}(Q_s, R) \quad (1 \le s \le 22). \tag{2.5}$$

We shall show that

$$\sum_{M_1} \dots \sum_{M_{22}} r(n;\mathbf{M}) \gg n^3, \tag{2.6}$$

where the multiple sum is over all choices of M_s of the form

$$M_s = 2^u P^\theta, \tag{2.7}$$

and satisfying (2.3). Since $p_s > R$, we find in the usual way that the verification of (2.6) is sufficient to establish that the number of representations of a large natural number n as the sum of 24 sixth powers is $\gg n^3$, and in particular shows that $G(6) \le 24$.

We henceforth assume that the M_s are of the form (2.7). Let

$$H(\alpha; L) = \sum_{1 \le x \le L} e(\alpha x^6), \quad g_s(\alpha) = \sum_{x \in \mathcal{A}(Q_s,R)} e(\alpha x^6), \tag{2.8}$$

and for convenience, write $F(\alpha) = H(\alpha; P)$. Then

$$r(n;\mathbf{M}) = \int_0^1 \mathcal{F}(\alpha) e(-\alpha n) d\alpha, \tag{2.9}$$

where

$$\mathcal{F}(\alpha) = F(\alpha)^2 \prod_{s=1}^{22} \left(\sum_{p_s} g_s(\alpha p_s^6) \right), \tag{2.10}$$

and the summation is over prime numbers p_s satisfying (2.4).

We shall require suitable estimates for certain auxiliary mean values. Let

$$\mathcal{I}_{1,s} = \int_0^1 \left| \sum_{p_s} g_s(\alpha p_s^6) \right|^{22} d\alpha, \quad \text{and} \quad \mathcal{I}_{2,s} = \int_0^1 \left(\sum_{p_s} |g_s(\alpha p_s^6)| \right)^{22} d\alpha,$$

where the summations are over the primes p_s satisfying (2.4). Also, write

$$\lambda_{11}^+ = 16.0994736 \quad \text{and} \quad \sigma = \tfrac{1}{32}. \tag{2.11}$$

Lemma 2.1. *The integrals $\mathcal{I}_{j,s}$ satisfy*

$$\mathcal{I}_{1,s} \ll P^{\lambda_{11}^+ + \varepsilon} \quad \textit{and} \quad \mathcal{I}_{2,s} \ll M_s P^{\lambda_{11}^+ + \varepsilon}.$$

Proof. By the argument of the proof of [**4**, Lemma 3.1] (see, in particular, equations (3.3), (3.7) and (3.11) of [**4**]), we have

$$\mathcal{I}_{1,s} \ll M_s^{19} \left(\left(PM_s + PH_sM_s(PM_s)^{-2\sigma} \right) Q_s^{\lambda_{10}+\varepsilon} + P^{1+\varepsilon} H_s M_s Q_s^{\frac{10}{11}\lambda_{11} - \frac{6}{11}} \right).$$

In view of (2.2), the bound for $\mathcal{I}_{1,s}$ now follows with a modicum of computation. Meanwhile, the argument of the proof of [**4**, Lemma 3.2] yields the desired bound for $\mathcal{I}_{2,s}$ in like manner.

3. Minor arcs with one difference

We now write $M = P^{\theta+\tau}$ and $Q = PM^{-1}$. Let $\mathfrak{m}$ denote the set of real numbers α in

$$\left(Q^{-6(1-\sigma)}, 1 + Q^{-6(1-\sigma)}\right]$$

with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$ and $|\alpha q - a| \le Q^{-6(1-\sigma)}$, one has $q > Q^{6\sigma} M^6$. Plainly we have $\mathfrak{m} \subset (0, 1)$.

Let $\mathfrak{n}$ denote the set of real numbers α in $(0, 1)$ with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a, q) = 1$ and $|\alpha q - a| \le Q^{-6(1-\sigma)}$ one has $q > 2^{-6} Q^{6\sigma}$. Write $r = p_s^6$, and define

$$G_{1,s}(\alpha) = \sum_{p_s} \sum_{1 \le h \le P/r} \sum_{\substack{hr < z \le 2P - hr \\ z \equiv h \pmod 2}} e\left(2^{-6}\alpha\Psi(z, h, p_s)\right),$$

where the first summation is over p_s satisfying (2.4), and where

$$\Psi(z, h, m) = m^{-6}\left((z + hm^6)^6 - (z - hm^6)^6\right).$$

Also, when $\mathcal{B} \subseteq (0, 1)$, write

$$\mathcal{J}_s(\mathcal{B}) = \int_{\mathcal{B}} \left|G_{1,s}(\alpha) g_s(\alpha)^{22}\right| d\alpha.$$

Lemma 3.1. *Let $\mathcal{F}(\alpha)$ be given by* (2.10)*. Then*

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)| d\alpha \ll \prod_{s=1}^{22} \left(P^{18-\nu} + M_s^{21} \mathcal{J}_s(\mathfrak{n})\right)^{\frac{1}{22}}.$$

Proof. Using the same argument as that in [**4**, section 4] (see, in particular, (4.9), (4.11) and (4.14) of [**4**]), we have

$$\int_{\mathfrak{m}} |\mathcal{F}(\alpha)| d\alpha \ll \prod_{s=1}^{22} \left(M_s^{21} I_s + J_s\right)^{\frac{1}{22}}, \tag{3.1}$$

where

$$I_s = P M_s Q_s^{\lambda_{11}+\varepsilon} + \int_{\mathfrak{n}} \left|G_{1,s}(\alpha) g_s(\alpha)^{22}\right| d\alpha, \tag{3.2}$$

$$J_s = \int_{\mathfrak{m}} \left(\sum_{p_s} \left|H(\alpha p_s^6; P/p_s)\right|^{\frac{1}{11}} \left|g_s(\alpha p_s^6)\right|\right)^{22} d\alpha,$$

and the summation in the second equation is over p_s satisfying (2.4).

First consider the integral J_s. Suppose that $\alpha \in \mathfrak{m}$, and choose $b \in \mathbb{Z}$ and $t \in \mathbb{Z}$ with

$$(b, t) = 1, \quad 1 \le t \le Q_s^5, \quad \text{and} \quad |\alpha p_s^6 t - b| \le Q_s^{-5}.$$

Then by Weyl's inequality and a standard major arc estimate (see [**2**, Theorem 4.1]), when $t > 2^{-6} Q_s^{6\sigma}$ or $|\alpha p_s^6 t - b| > Q_s^{-6(1-\sigma)}$ we have

$$H(\alpha p_s^6; P/p_s) \ll Q_s^{1-\sigma+\varepsilon}. \tag{3.3}$$

Otherwise, $t p_s^6 \le Q^{6\sigma} M^6$ and $|\alpha p_s^6 t - b| \le Q^{-6(1-\sigma)}$, and hence $\alpha \notin \mathfrak{m}$. So we may conclude that (3.3) holds uniformly on $\mathfrak{m}$, and hence, by Lemma 2.1, that

$$J_s \ll Q_s^{2-2\sigma+\varepsilon} \mathcal{I}_{2,s} \ll Q_s^{2-2\sigma} M_s P^{\lambda_{11}^{+}+\varepsilon}.$$

Thus a little computation reveals that

$$J_s \ll P^{18-\delta}. \tag{3.4}$$

Further, on recalling (2.1) we find that

$$PM_s^{22}Q_s^{\lambda_{11}+\varepsilon} \ll P^{\lambda_{12}+(\lambda_{11}+\varepsilon-22)\nu} \ll P^{18-\nu}.$$

The lemma therefore follows on combining (3.1), (3.2) and (3.4).

We proceed by pruning the minor arcs $\mathfrak{n}$. Write $M' = P^{\phi}$, $Q'_s = Q_sM'^{-1}$, and $Q' = QM'^{-1}$. We take $\mathfrak{p}$ to be the set of real numbers α in $\mathfrak{n}$ with the property that whenever $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q)=1$ and $|\alpha q - a| \le P^{\delta}Q'^{-6}$ one has $q > P^{\delta}M'^{6}$. We then take $\mathfrak{P} = \mathfrak{n}\backslash\mathfrak{p}$.

We shall require a suitable major arc estimate for $G_{1,s}(\alpha)$. When $(a,q)=1$, we take $\mathfrak{P}(q,a)$ to be the set of α in $\mathfrak{n}$ for which $P^{-\delta}Q'^{6}|q\alpha - a| \le 1$. We then define $G^*(\alpha)$ to be

$$q^{\varepsilon}PH_sM_s(q+Q_s^6|q\alpha-a|)^{-1/5}$$

when $\alpha \in \mathfrak{P}(q,a)$ and $0 \le a \le q \le P^{\delta}M'^{6}$, and to be zero otherwise.

Lemma 3.2. *Suppose that* $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $(a,q)=1$, *and*

$$Q_s^6|q\alpha - a| \le P^{1-\delta}.$$

Then we have

$$G_{1,s}(\alpha) \ll G^*(\alpha) + H_sM_sq^{\frac{4}{5}+\varepsilon}.$$

Proof. This follows from [**4**, Lemma 4.3].

We now show that the contribution from the arcs $\mathfrak{P}$ is negligible.

Lemma 3.3. *Suppose that* $1 \le s \le 22$. *Then*

$$M_s^{21}\mathcal{J}_s(\mathfrak{n}) \ll P^{18-\nu} + M_s^{21}\mathcal{J}_s(\mathfrak{p}).$$

Proof. Since $\mathfrak{n} = \mathfrak{P}\cup\mathfrak{p}$, we have merely to obtain a suitable bound for $\mathcal{J}_s(\mathfrak{P})$. Let $\alpha \in \mathfrak{P}$. On noting that $\alpha \notin \mathfrak{p}$, it follows from an application of Dirichlet's theorem that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with

$$(a,q)=1, \quad |\alpha q - a| \le P^{\delta}Q'^{-6} \quad \text{and} \quad q \le P^{\delta}M'^{6}.$$

Then

$$Q_s^6|q\alpha - a| \le P^{2\delta}M'^{6} \le P^{1-\delta},$$

and hence we may apply Lemma 3.2 to deduce that

$$G_{1,s}(\alpha) \ll G^*(\alpha) + H_sM_s\left(P^{\delta}M'^{6}\right)^{\frac{4}{5}+\varepsilon}.$$

Thus

$$\mathcal{J}_s(\mathfrak{P}) \ll \mathcal{T}_1 + \mathcal{T}_2, \tag{3.5}$$

where

$$\mathcal{T}_1 = \left(P^{\delta}M'^{6}\right)^{\frac{4}{5}+\varepsilon}H_sM_s\int_0^1|g_s(\alpha)|^{22}d\alpha,$$

and

$$\mathcal{T}_2 = \int_{\mathfrak{P}} G^*(\alpha)|g_s(\alpha)|^{22}d\alpha.$$

On using (2.1), a little computation shows that

$$M_s^{21}\mathcal{T}_1 \ll P^{\varepsilon}\left(P^{\delta}M'^{6}\right)^{4/5}H_sM_s^{22}Q_s^{\lambda_{11}} \ll P^{18-\delta}. \tag{3.6}$$

Next, by Hölder's inequality,

$$\mathcal{T}_2 \ll \left(\int_{\mathfrak{P}} G^*(\alpha)^{12}d\alpha\right)^{\frac{1}{12}}\left(\int_0^1|g_s(\alpha)|^{24}d\alpha\right)^{\frac{11}{12}}.$$

It follows from (1.1) that the last integral is $\ll Q_s^{18+\varepsilon}$. Further, since $\alpha \in \mathfrak{n}$, we have either $|\alpha q - a| > Q^{-6(1-\sigma)}$ or $q > 2^{-6}Q^{6\sigma}$. A straightforward estimation therefore gives

$$\int_{\mathfrak{P}} G^*(\alpha)^{12}d\alpha \ll (P^{1-\delta}H_sM_s)^{12}Q_s^{-6}.$$

Thus

$$M_s^{21}\mathcal{T}_2 \ll P^{1-\delta}H_sM_s^{22}Q_s^{16} \ll P^{18-\nu}. \tag{3.7}$$

The lemma now follows on collecting together (3.5), (3.6) and (3.7).

4. Minor arcs with two differences: a fundamental lemma

Our next step is to take a further efficient difference on the minor arcs $\mathfrak{p}$. However, before we proceed we record an estimate for the mean value

$$\mathcal{K}_s = \int_0^1 |G_{1,s}(\alpha)^2 g_s(\alpha)^{20}| d\alpha.$$

Lemma 4.1. *Suppose that* $1 \le s \le 22$. *Then*

$$\mathcal{K}_s \ll P^{18+\varepsilon} M_s^{-24}.$$

Proof. By considering the underlying diophantine equations, we find that $\mathcal{K}_s$ is bounded above by the number of solutions of the equation

$$\Psi(z_1, h_1, p_1) - \Psi(z_2, h_2, p_2) = \sum_{i=1}^{10} (x_i^6 - y_i^6), \tag{4.1}$$

with p_i $(i = 1, 2)$ prime numbers satisfying $M_s < p_i \le 2M_s$ $(i = 1, 2)$, and with

$$1 \le h_i \le P p_i^{-6}, \quad 1 \le z_i \le 2P \quad (i = 1, 2),$$

and

$$x_i, y_i \in \mathcal{A}(Q_s, R) \quad (1 \le i \le 10).$$

We observe that the equation (4.1) is in a suitable form for the repeated efficient differencing process of [**6**] to be applied (compare the above equation with [**6**, equation (2.4)]). Before extracting a further efficient difference, we shall require some additional notation. We define $\Psi_2(z, \mathbf{h}, \mathbf{m})$ by

$$\Psi_2(z, \mathbf{h}, \mathbf{m}) = m_2^{-6} \left(\Psi(z + h_2 m_2^6, h_1, m_1) - \Psi(z - h_2 m_2^6, h_1, m_1) \right). \tag{4.2}$$

Also, we write $H' = PM'^{-6}$, and

$$K(\alpha) = \sum_{1 \le h_1 \le H_s} \sum_{1 \le h_2 \le H'} \sum_{\substack{M_s < m_1 \le 2M_s \\ m_1 \text{ prime}}} \sum_{\substack{M' < m_2 \le M'R \\ m_2 \in \mathcal{A}(P,R)}} \sum_{1 \le z \le 2P} e(\alpha \Psi_2(z, \mathbf{h}, \mathbf{m})).$$

Now, by applying the methods which underly the proof of [**5**, Lemma 2.1] (see Lemmata 2.2, 2.3 and 3.1 of [**6**]), we find that

$$\mathcal{K}_s \ll P^\varepsilon M_s H_s M'^{19} \left(PM' M_s H_s {Q'_s}^{\lambda_{10}} + T \right),$$

where

$$T = \int_0^1 |K(\alpha) f_2(\alpha)^{20}| d\alpha,$$

and

$$f_2(\alpha) = \sum_{x \in \mathcal{A}(Q'_s, R)} e(\alpha x^6).$$

The integral T may be estimated through a Hardy-Littlewood dissection, and indeed such a procedure is executed in [**5**, Lemma 12.4]. Thus, on applying the latter lemma with $t = 28$, we obtain

$$T \ll P^{1+\varepsilon} M_s M' H_s H' \left(Z^{-1/8} {Q'_s}^{\lambda_{10}} + {Q'_s}^{14} \right),$$

where

$$Z = P M_s^{\frac{27}{28}} \left(P^{1/3} M'^{44-\mu_{28}} \right)^{\frac{1}{28}},$$

and μ_{28} is any exponent for which we have the estimate

$$\int_0^1 \Bigg| \sum_{x \in \mathcal{A}(Q_s', R)} e(\alpha x^{12}) \Bigg|^{56} d\alpha \ll {Q_s'}^{\mu_{28}+\varepsilon}.$$

An application of [**6**, Lemma 3.2] shows that $\mu_{28} = 44.2211063$ is permissible. Thus, after a little calculation, we deduce that

$$T \ll P^{1+\varepsilon} M_s M' H_s H' {Q_s'}^{14},$$

and hence

$$\mathcal{K}_s \ll P^{\varepsilon} \left(P{M'}^{20}(M_s H_s)^2 {Q_s'}^{\lambda_{10}} + P{M'}^{20}(M_s H_s)^2 H' {Q_s'}^{14} \right).$$

Now observe that the definition of ϕ_{12} given in section 2 implies that

$$\begin{aligned} \Delta_{10}^*(1-\theta-\phi) - 1 + 6\phi &= (6-\Delta_{10}^*)(\phi_{12}-\nu) - 1 + \Delta_{10}^*(1-\theta_{12}+\nu) \\ &= -(6-2\Delta_{10}^*)\nu. \end{aligned}$$

Thus ${Q_s'}^{\lambda_{10}} \ll H' {Q_s'}^{14}$, and so

$$\mathcal{K}_s \ll P^{1+\varepsilon} {M'}^{20} (M_s H_s)^2 H' {Q_s'}^{14}.$$

The lemma now follows on recalling the definitions of H', H_s and Q_s'.

Having prepared the ground we now turn our attention to the main task of developing a variant of the iterative method restricted to minor arcs. In principle we imitate the proof of [**3**, Lemma 2.1]. It might be thought that the proof of [**6**, Lemma 2.2] would prove more readily adaptable to the task at hand. However, the former permits greater control to be exercised on the minor arcs. To facilitate our analysis, we write

$$f_1^*(\alpha) = \sum_{\substack{x \in \mathcal{A}(Q_s, R) \\ x > M'}} e(\alpha x^6), \quad f(\alpha; L) = \sum_{x \in \mathcal{A}(L,R)} e(\alpha x^6).$$

Then by using the combinatorial lemma [**3**, Lemma 10.1], we deduce that

$$f_1^*(\alpha) = \sum_{\pi \le R} \sum_{u \in \mathcal{B}(M', \pi, R)} \sum_{v \in \mathcal{A}(Q_s/u, \pi)} e(\alpha u^6 v^6),$$

where π takes prime values, and

$$\mathcal{B}(L, \pi, R) = \{x \in \mathbb{N} \ : \ L < x \le L\pi, \ \pi | x, \ \text{and } p|x \Rightarrow \pi \le p \le R\}.$$

We have

$$\mathcal{J}_s(\mathfrak{p}) \ll V_1 + V_2, \tag{4.3}$$

where

$$V_1 = \int_0^1 \left| G_{1,s}(\alpha) g_s(\alpha)^{21} f(\alpha; M') \right| d\alpha,$$

and

$$V_2 = \int_{\mathfrak{p}} \left| G_{1,s}(\alpha) g_s(\alpha)^{21} f_1^*(\alpha) \right| d\alpha. \tag{4.4}$$

The integral V_1 may be disposed of swiftly by using Hölder's inequality. We obtain

$$V_1 \le \left(\int_0^1 |G_{1,s}(\alpha)^2 g_s(\alpha)^{20}| d\alpha \right)^{1/2} \left(\int_0^1 |g_s(\alpha)|^{24} d\alpha \right)^{\frac{11}{24}} \left(\int_0^1 |f(\alpha; M')|^{24} d\alpha \right)^{\frac{1}{24}},$$

so that by Lemma 4.1 combined with (1.1),

$$V_1 \ll \left(P^{18+\varepsilon} M_s^{-24} \right)^{1/2} \left(Q_s^{18+\varepsilon} \right)^{\frac{11}{24}} \left({M'}^{18+\varepsilon} \right)^{\frac{1}{24}} \ll P^{18-\delta} M_s^{-21}. \tag{4.5}$$

As a first step in treating V_2 we turn our attention to the singular solutions present in our underlying equation. To this end we write $D = P^\gamma$ and

$$G_{1,s}(\alpha)f_1^*(\alpha) = \mathcal{G}(\alpha) + \mathcal{H}(\alpha), \tag{4.6}$$

where

$$\mathcal{G}(\alpha) = \sum_\pi \sum_u \sum_{(\Psi',u)>D} \sum_v e\left(2^{-6}\alpha\Psi + \alpha u^6 v^6\right),$$

and

$$\mathcal{H}(\alpha) = \sum_\pi \sum_u \sum_{(\Psi',u)\le D} \sum_v e\left(2^{-6}\alpha\Psi + \alpha u^6 v^6\right).$$

Here and throughout this section (unless explicitly indicated otherwise) we shall adopt the following convention concerning summation signs involving the symbols π, u, v and Ψ. The appearance of π will indicate a summation over primes $\pi \le R$, u will indicate a summation over $u \in \mathcal{B}(M', \pi, R)$, v will indicate a summation over $v \in \mathcal{A}(Q_s/u, \pi)$, and Ψ will indicate a summation over p_s, z, h satisfying (2.4), $r = p_s^6$, $1 \le h \le P/r$, $hr < z \le 2P - hr$, and $z \equiv h \pmod 2$.

Collecting together (4.3)-(4.6), we obtain the following bound on $\mathcal{J}_s(\mathfrak{p})$.

Lemma 4.2. *The integral $\mathcal{J}_s(\mathfrak{p})$ satisfies*

$$\mathcal{J}_s(\mathfrak{p}) \ll P^{18-\delta} M_s^{-21} + \int_0^1 \left|\mathcal{G}(\alpha) g_s(\alpha)^{21}\right| d\alpha + \int_{\mathfrak{p}} \left|\mathcal{H}(\alpha) g_s(\alpha)^{21}\right| d\alpha.$$

We first estimate the integral in which $\mathcal{G}(\alpha)$ appears.

Lemma 4.3. *Let $\mathcal{G}(\alpha)$ be defined as above. Then*

$$\int_0^1 \left|\mathcal{G}(\alpha) g_s(\alpha)^{21}\right| d\alpha \ll P^{18-\varepsilon} M_s^{-21}.$$

Proof. We first observe that

$$\begin{aligned}\mathcal{G}(\alpha) &= \sum_{D<d\le M'R} \sum_\pi \sum_{d|u} \sum_{(\Psi',u)=d} \sum_v e\left(2^{-6}\alpha\Psi + \alpha u^6 v^6\right) \\ &= \sum_{D<d\le M'R} \sum_\pi \sum_{d|u} \sum_{d|\Psi'} \sum_{e|(\frac{\Psi'}{d}, \frac{u}{d})} \mu(e) \sum_v e\left(2^{-6}\alpha\Psi + \alpha u^6 v^6\right).\end{aligned}$$

Thus

$$\begin{aligned}\mathcal{G}(\alpha) &= \sum_{D<c\le M'R} \sum_{\substack{ed=c\\ d>D}} \sum_\pi \sum_{c|u} \sum_{c|\Psi'} \mu(e) \sum_v e\left(2^{-6}\alpha\Psi + \alpha u^6 v^6\right) \\ &= \sum_{D<c\le M'R} \lambda(c) G_c(\alpha) f_c(\alpha),\end{aligned}$$

where

$$\lambda(c) = \sum_{\substack{ed=c\\ d>D}} \mu(e), \quad G_c(\alpha) = \sum_{c|\Psi'} e\left(2^{-6}\alpha\Psi\right),$$

and

$$f_c(\alpha) = \sum_\pi \sum_{c|u} \sum_v e\left(\alpha u^6 v^6\right).$$

Thus, by Hölder's inequality,

$$\begin{aligned}\int_0^1 \left|\mathcal{G}(\alpha) g_s(\alpha)^{21}\right| d\alpha &\le \sum_{D<c\le M'R} |\lambda(c)| \int_0^1 \left|G_c(\alpha) f_c(\alpha) g_s(\alpha)^{21}\right| d\alpha \\ &\ll A(D)^{1/2} B(D)^{1/2},\end{aligned} \tag{4.7}$$

where

$$A(D) = \sum_{D<c\le M'R} |\lambda(c)| \int_0^1 \left| G_c(\alpha)^2 g_s(\alpha)^{20} \right| d\alpha \tag{4.8}$$

and

$$B(D) = \sum_{D<c\le M'R} |\lambda(c)| \left(\int_0^1 |f_c(\alpha)|^{24} d\alpha \right)^{\frac{1}{12}} \left(\int_0^1 |g_s(\alpha)|^{24} d\alpha \right)^{\frac{11}{12}}. \tag{4.9}$$

First observe that when $D < c \le M'R$, we have

$$|\lambda(c)| \le \sum_{e|c} 1 \ll P^{\varepsilon}. \tag{4.10}$$

Further, by considering the underlying diophantine equation, it follows that the integral in (4.8) is equal to the number of solutions of the equation

$$\Psi(z,h,p) - \Psi(z',h',p') = \sum_{i=1}^{10} \left((2x_i)^6 - (2y_i)^6 \right)$$

with z, h, p, and likewise z', h', p', in the ranges indicated in the discussion preceding Lemma 4.2, and also satisfying $c|\Psi'(z,h,p)$ and $c|\Psi'(z',h',p')$, and with $x_i, y_i \in \mathcal{A}(Q_s, R)$ $(1 \le i \le 10)$. Then by combining standard estimates for the divisor function with Lemma 4.1 and (4.10), we deduce that

$$A(D) \ll P^{18+\varepsilon} M_s^{-24}. \tag{4.11}$$

In order to dispose of $B(D)$, we first note that in the definition of $f_c(\alpha)$, the summation conditions on π, u and v imply that each product uv which occurs is unique. Thus, by considering the underlying diophantine equation,

$$\int_0^1 |f_c(\alpha)|^{24} d\alpha \le B_c, \tag{4.12}$$

where B_c denotes the number of solutions of the equation

$$\sum_{i=1}^{12} \left(x_i^6 - y_i^6 \right) = 0$$

with $x_i, y_i \in \mathcal{A}(Q_s, R)$, $c|x_i$ and $c|y_i$. Therefore, by (1.1) we have

$$B_c \ll (Q_s/c)^{18+\varepsilon}.$$

Also by (1.1),

$$\int_0^1 |g_s(\alpha)|^{24} d\alpha \ll Q_s^{18+\varepsilon},$$

and so by (4.9), (4.10) and (4.12),

$$\begin{aligned} B(D) &\ll \sum_{D<c\le M'R} P^{\varepsilon} B_c^{1/12} \left(Q_s^{18+\varepsilon} \right)^{11/12} \\ &\ll Q_s^{18+\varepsilon} \sum_{D<c\le M'R} c^{-3/2} \\ &\ll Q_s^{18+\varepsilon} D^{-1/2}. \end{aligned} \tag{4.13}$$

On combining (4.7), (4.11) and (4.13), we deduce that

$$\int_0^1 \left| \mathcal{G}(\alpha) g_s(\alpha)^{21} \right| d\alpha \ll P^{18+\varepsilon} M_s^{-21} D^{-1/4},$$

and the lemma follows in view of our hierarchy Ω.

We now turn our attention to $\mathcal{H}(\alpha)$, which we write in the form

$$\mathcal{H}(\alpha)=\sum_{d\le D}\sum_{\pi}\sum_{d|u} h_1(\alpha)G_2(\alpha;d,u),$$

where

$$h_1(\alpha)=\sum_{v\in\mathcal{A}(Q_s/u,\pi)} e\left(\alpha u^6 v^6\right),$$

and

$$G_2(\alpha;d,w)=\sum_{(\Psi',w)=d} e\left(2^{-6}\alpha\Psi\right).$$

On writing

$$h_\pi(\alpha,\theta)=\sum_{v\in\mathcal{A}(Q_s',\pi)} e(\alpha v^6+\theta v),$$

we deduce that

$$h_1(\alpha)=\int_0^1 h_\pi(\alpha u^6,\theta)\sum_{1\le w\le Q_s/u} e(-\theta w)d\theta.$$

Therefore

$$\int_{\mathfrak{p}}|\mathcal{H}(\alpha)g_s(\alpha)^{21}|d\alpha$$
$$\le\sum_{d\le D}\sum_{\pi}\sum_{d|u}\int_{\mathfrak{p}}\left|G_2\left(\alpha;d,u\right)g_s(\alpha)^{21}\right|\int_0^1\left|h_\pi(\alpha u^6,\theta)\right|\Upsilon(\theta)d\theta d\alpha,$$

where

$$\Upsilon(\theta)=\min\{Q_s',\|\theta\|^{-1}\}.$$

Then

$$\int_{\mathfrak{p}}\left|\mathcal{H}(\alpha)g_s(\alpha)^{21}\right|d\alpha\le\int_0^1\sum_{d\le D}\sum_{\pi}\sum_{d|u}\Upsilon(\theta)J(d,\pi,u,\theta)d\theta, \tag{4.14}$$

where

$$J(d,\pi,u,\theta)=\int_{\mathfrak{p}}\left|G_2\left(\alpha;d,u\right)g_s(\alpha)^{21}h_\pi(\alpha u^6,\theta)\right|d\alpha. \tag{4.15}$$

On applying Hölder's inequality first to (4.15), and then to (4.14), we may conclude as follows.

Lemma 4.4. *For $i=1,2,3$ let*

$$K_i=\int_0^1\sum_{d\le D}\sum_{\pi}\sum_{d|u} L_i\Upsilon(\theta)d\theta,$$

where

$$L_1=L_1(u,d,\pi,\theta)=\int_{\mathfrak{p}}\left|G_2\left(\alpha;d,u\right)^2 h_\pi(\alpha u^6,\theta)^{20}\right|d\alpha,$$

$$L_2=L_2(u,d)=\int_0^1\left|G_2\left(\alpha;d,u\right)^2 g_s(\alpha)^{20}\right|d\alpha,$$

$$L_3=\int_0^1|g_s(\alpha)|^{24}d\alpha.$$

Then

$$\int_{\mathfrak{p}}\left|\mathcal{H}(\alpha)g_s(\alpha)^{21}\right|d\alpha\le K_1^{\frac{1}{20}}K_2^{\frac{9}{20}}K_3^{\frac{1}{2}}.$$

We are able to dismiss K_2 and K_3 rapidly with the following lemmata.

Lemma 4.5. *Let K_3 be as above. Then $K_3 \ll M'Q_s^{18+\varepsilon}$.*

Proof. By (1.1), we have $L_3 \ll Q_s^{18+\varepsilon}$. The desired conclusion follows on noting that

$$\sum_{d\le D}\sum_{\pi}\sum_{d|u} 1 \le \sum_{d\le D}\sum_{\pi} M'R/d \ll M'R^2D^\varepsilon \ll M'Q_s^\varepsilon.$$

Lemma 4.6. *Let K_2 be as above. Then $K_2 \ll M'P^{18+\varepsilon}M_s^{-24}$.*

Proof. By considering the underlying diophantine equations, from Lemma 4.1 we have

$$L_2(u,d) \ll P^{18+\varepsilon}M_s^{-24}.$$

The lemma now follows almost immediately, as in the proof of the previous lemma.

5. Minor arcs with two differences: Efficient differencing.

It remains to consider K_1, which in turn depends on $L_1(u,d,\pi,\theta)$. We are able to deal with the latter through an argument similar to the one used in [**4**, section 4]. Let $\mathfrak{q}$ denote the set of real numbers α in $(0,1]$ with the property that whenever $a\in\mathbb{Z}$, $q\in\mathbb{N}$, $(a,q)=1$ and $|\alpha q - a| \le P^\delta Q'^{-6}$, then one has $q > P^\delta R^{-6}$. Now put

$$\mathfrak{q}_c = \{\alpha : \alpha - c \in \mathfrak{q}\},$$

and for a given integer u with $M' < u \le M'R$, write

$$w = u^6, \quad \mathcal{B}_w = \bigcup_{c=0}^{w-1} \mathfrak{q}_c, \quad \mathcal{C}_w = \{\alpha : \alpha w \in \mathcal{B}_w\}.$$

In this section and those following, we shall adopt the convention that $\sum_{h,m}$ denotes a summation over those integers h and m satisfying

$$M_s < m \le 2M_s \quad \text{and} \quad 1 \le h \le H_s,$$

with obvious modifications where appropriate. Also, in this section we shall write $\sum_y$ for a summation over integers y satisfying

$$hm^6 + 32gu^6 < y \le 2P - hm^6 - 32gu^6 \quad \text{and} \quad y \equiv h \pmod 2.$$

Lemma 5.1. *With Ψ_2 defined by equation (4.2), write*

$$G_3(\alpha;u) = \sum_{h,m}\sum_{1\le g\le H'}\left|\sum_y (\alpha 2^{-6}\Psi_2(2y;2h,g;m,2u))\right|.$$

Then

$$L_1(u,d,\pi,\theta) \ll P^\varepsilon H_s M_s D^6 \mathcal{Z},$$

where

$$\mathcal{Z} = PH_sM_s\int_0^1 |h_\pi(\alpha,\theta)|^{20}\,d\alpha + \int_{\mathfrak{q}} \left|G_3(\alpha;u)h_\pi(\alpha,\theta)^{20}\right|\,d\alpha.$$

Proof. By the argument of [**4**, section 4] we have $\mathfrak{p}\subset\mathcal{C}_w$. As in [**4**, section 4] it now follows that

$$L_1(u,d,\pi,\theta) \le \int_{\mathfrak{q}} H(\alpha)\,|h_\pi(\alpha,\theta)|^{20}\,d\alpha, \tag{5.1}$$

where

$$H(\alpha) = w^{-1}\sum_{c=0}^{w-1}\left|G_2\left(\frac{\alpha+c}{w};d,u\right)\right|^2.$$

Let $t = 2^6 w = (2u)^6$. Then by Cauchy's inequality, we have

$$H(\alpha) \ll H_s M_s \sum_{h,m} w^{-1} \sum_{c=0}^{w-1} |H_1(\alpha; c)|^2,$$

where

$$H_1(\alpha; c) = \sum_{\substack{x=1 \\ (\Psi'(x,h,m),u)=d}}^{t} e\left(\frac{c}{t}\Psi(x,h,m)\right) \sum_{z} e\left(\frac{\alpha}{t}\Psi(z,h,m)\right),$$

the final summation being over integers z satisfying

$$hm^6 < z \le 2P - hm^6, \quad z \equiv x \pmod{t} \quad \text{and} \quad z \equiv h \pmod{2}. \tag{5.2}$$

Therefore, by orthogonality,

$$H(\alpha) \ll H_s M_s \sum_{h,m} \sum_{z,z'} e\left(\frac{\alpha}{t}\left(\Psi(z',h,m) - \Psi(z,h,m)\right)\right), \tag{5.3}$$

where the final summation is over integers z and z' satisfying

$$hm^6 < z, z' \le 2P - hm^6, \tag{5.4}$$

$$z' \equiv z \equiv h \pmod{2}, \tag{5.5}$$

$$\Psi(z,h,m) \equiv \Psi(z',h,m) \pmod{t}, \quad (\Psi'(z,h,m),u) = (\Psi'(z',h,m),u) = d.$$

Let $\mathcal{D}(b; h, m)$ denote the set of integers x satisfying the conditions

$$1 \le x \le t, \quad \Psi(x,h,m) \equiv b \pmod{t} \quad \text{and} \quad (\Psi'(x,h,m),u) = d.$$

Then by classifying the congruence class of $\Psi(z,h,m)$ and $\Psi(z',h,m)$ modulo t, we may rewrite (5.3) to obtain

$$H(\alpha) \ll H_s M_s \sum_{h,m} \sum_{b=1}^{t} \left| \sum_{x \in \mathcal{D}(b;h,m)} \sum_{z} e\left(\frac{\alpha}{t}\Psi(z,h,m)\right)\right|^2,$$

where the last summation is over integers z satisfying (5.2). A simple estimation gives

$$\text{card}(\mathcal{D}(b;h,m)) \ll d^6 t^{\varepsilon},$$

and hence by Cauchy's inequality,

$$H(\alpha) \ll H_s M_s \sum_{h,m} d^6 P^{\varepsilon} \sum_{x=1}^{t} \left| \sum_{z} e\left(\frac{\alpha}{t}\Psi(z,h,m)\right)\right|^2,$$

where the last summation is again over integers z satisfying (5.2). Therefore

$$H(\alpha) \ll P^{\varepsilon} H_s M_s d^6 \sum_{h,m} \sum_{z,z'} e\left(\frac{\alpha}{t}\left(\Psi(z',h,m) - \Psi(z,h,m)\right)\right),$$

where the summation over z, z' satisfies (5.4), (5.5), and $z \equiv z' \pmod{t}$. On isolating the diagonal terms, we therefore deduce that

$$H(\alpha) \ll P^{1+\varepsilon}(H_s M_s)^2 d^6 + P^{\varepsilon} H_s M_s d^6 |G_3'(\alpha; u)|, \tag{5.6}$$

where

$$G_3'(\alpha; u) = \sum_{h,m} \sum_{z,z'} e\left(\frac{\alpha}{t}\left(\Psi(z',h,m) - \Psi(z,h,m)\right)\right),$$

and the second summation is over integers z, z' satisfying (5.5) and

$$hm^6 < z < z' \leq 2P - hm^6, \quad z \equiv z' \pmod{t}. \tag{5.7}$$

Now write $\mathcal{E}(h,m)$ for the set of integers g with $1 \leq g \leq 2P/t$ such that for some integers z and z' satisfying (5.5) and (5.7), we have $z' - z = gt$. For each pair of integers z and z' satisfying (5.5) and (5.7), we define the integers y and g by $2y = z' + z$ and $gt = z' - z$. With this change of variables, we find that

$$G_3'(\alpha; u) = \sum_{h,m} \sum_{g \in \mathcal{E}(h,m)} \sum_y e\left(\alpha \Phi_2(y; h, g; m, u)\right),$$

where, on recalling (4.2),

$$\begin{aligned}\Phi_2(y; h, g; m, u) &= t^{-1}\left(\Psi(y + \tfrac{1}{2}gt; h, m) - \Psi(y - \tfrac{1}{2}gt; h, m)\right)\\ &= 2^{-6}\Psi_2(2y; 2h, g; m, 2u).\end{aligned}$$

The lemma now follows from (5.1) and (5.6).

Recalling the statement of Lemma 4.4, from Lemma 5.1 we deduce that

$$K_1 \ll U_1 + P^\varepsilon H_s M_s D^6 \int_0^1 \sum_{d \leq D} \sum_\pi U_2(\theta)\Upsilon(\theta) d\theta,$$

where

$$U_1 = P^{1+\varepsilon}(H_s M_s)^2 D^6 \sum_{d \leq D} \sum_\pi \sum_{d|u} \int_0^1 \Upsilon(\theta) \int_0^1 |h_\pi(\alpha, \theta)|^{20} d\alpha d\theta,$$

and

$$U_2(\theta) = \int_{\mathfrak{q}} H_3(\alpha)|h_\pi(\alpha, \theta)|^{20} d\alpha.$$

Here, we have written

$$H_3(\alpha) = \sum_{\substack{M' < u \leq M'R \\ u \in \mathcal{A}(P,R)}} \sum_{h,m} \sum_{1 \leq g \leq H'} \left| \sum_y e(\alpha 2^{-6}\Psi_2(2y; 2h, g; m, 2u)) \right|. \tag{5.8}$$

The inner integral in U_1 counts the number of solutions of the diophantine equation

$$\sum_{i=1}^{10} (x_i^6 - y_i^6) = 0,$$

with $x_i, y_i \in \mathcal{A}(Q_s', \pi)$, and with each solution weighted by the factor

$$e(\theta(x_1 + \cdots + x_{10} - y_1 - \cdots - y_{10})).$$

Thus

$$\begin{aligned}U_1 &\ll P^{1+\varepsilon}(H_s M_s)^2 D^6 \sum_{d \leq D} \sum_\pi \sum_{d|u} S_{10}(Q_s', R)\\ &\ll P^{1+\varepsilon}(H_s M_s)^2 D^6 R^2 M' {Q_s'}^{\lambda_{10}}.\end{aligned}$$

We now note that in view of the discussion in section 2, we have

$$\begin{aligned}3 - 10(\theta_{12} - \nu) &+ 20(\phi_{12} - \nu) + \lambda_{10}(1 - \theta_{12} - \phi_{12} + 2\nu)\\ &= (2\lambda_{10} - 10)\nu + (6 - \Delta_{10}^*)\phi_{12} + 17 + \Delta_{10}^*(1 - \theta_{12}) - 24\theta_{12}\\ &= 18 - 24(\theta_{12} - \nu) + (2\lambda_{10} - 34)\nu.\end{aligned}$$

Therefore

$$U_1 \ll P^{\varepsilon + (2\lambda_{10} - 34)\nu} R^2 D^6 P^{18} M_s^{-24} {M'}^{-19}.$$

In view of our hierarchy Ω, we may therefore conclude this section with the following lemma.

Lemma 5.2. *Let K_1 be defined as in Lemma 4.4. Then*

$$K_1 \ll P^{18-\nu} M_s^{-24} {M'}^{-19} + P^\varepsilon H_s M_s D^6 \int_0^1 \sum_{d \leq D} \sum_\pi U_2(\theta)\Upsilon(\theta) d\theta.$$

6. Further pruning

We must now estimate $U_2(\theta)$. Let $\mathfrak{r}$ denote the set of real numbers α in $(0,1]$ with the property that whenever $a\in\mathbb{Z}$, $q\in\mathbb{N}$, $(a,q)=1$ and $|\alpha q-a|\le PQ'^{-6}R^{-24}$, one has $q>P$. Then $\mathfrak{r}\subset\mathfrak{q}$. Further, we write $\mathfrak{R}=\mathfrak{q}\setminus\mathfrak{r}$. We first establish a "major arc" estimate for $H_3(\alpha)$ to assist in our pruning procedures. Throughout this section we shall adopt the convention that $\sum_{g,u}$ denotes a summation over those integers g and u satisfying

$$1\le g\le H',\quad M'<u\le M'R,\quad u\in\mathcal{A}(P,R),$$

with obvious modifications where appropriate.

Lemma 6.1. *Suppose that $(a,q)=1$, $\beta=\alpha-a/q$, and $qP^{-1}{Q'_s}^6R^{24}|\beta|\le 1$. Then*

$$H_3(\alpha)\ll\sum_{m,u}\sum_{h,g}\frac{Pq^{-1}\tau(q,a,h,g,m,u)}{(1+|\beta|hgP^4)^{1/4}}+H_sH'M_sM'q^{\frac34+\varepsilon},$$

where

$$\tau(q,a,h,g,m,u)=\left|\sum_{r=1}^{q}e\left(\frac{a}{q}2^{-6}\Psi_2(2r;2h,g;m,2u\right)\right|.$$

Proof. We are able to use essentially the same analysis for the exponential sum in question as was used in the proof of [**5**, Lemma 4.7] for the sum $F_2(\alpha)$. The extra restrictions on the variable y are merely an inconvenience in the course of the argument.

When $(a,q)=1$, we take $\mathfrak{R}(q,a)$ to be the set of α in $\mathfrak{q}$ for which $qP^{-1}Q'^6R^{24}|\alpha-a/q|\le 1$. We then define $H_3^*(\alpha)$ to be

$$\sum_{m,u}\sum_{h,g}\frac{Pq^{-1}\tau(q,a,h,g,m,u)}{(1+|\alpha-a/q|hgP^4)^{1/4}}$$

when $\alpha\in\mathfrak{R}(q,a)$ and $0\le a\le q\le P$, and to be zero otherwise. We now derive an analogue of [**5**, Lemma 4.10].

Lemma 6.2. *Let $H_3^*(\alpha)$ be the function defined above. Then*

$$\int_{\mathfrak{R}}|H_3^*(\alpha)|^6d\alpha\ll P^{\varepsilon-\delta/4}(PH_sH'M_sM')^6{Q'_s}^{-6}.$$

Proof. Suppose that $(a,q)=1$ and $\alpha\in\mathfrak{R}(q,a)$. Then by [**5**, Lemma 4.8] (as in the proof of [**5**, Lemma 4.10]), we have

$$\tau(q,a,h,g,m,u)\ll q^{3/4+\varepsilon}(q,hg)^{1/4}.$$

Therefore, on writing $d=(q,hg)$, we deduce that

$$\begin{aligned}H_3^*(\alpha)&\ll q^{\varepsilon-1/4}\sum_{m,u}\sum_{d|q}d^{1/4}\min\left\{PH_sH'd^{-1},(H_sH')^{3/4}|\alpha-a/q|^{-1/4}d^{-1}\right\}\\&\ll q^{\varepsilon-1/4}PH_sH'M_sM'R\min\left\{1,({Q'_s}^6|\alpha-a/q|)^{-1/4}\right\}.\end{aligned}\tag{6.1}$$

Next, whenever $\alpha\in\mathfrak{R}(q,a)$ we have $\alpha\in\mathfrak{q}$, and hence either $q>P^\delta R^{-6}$ or $|\alpha q-a|>P^\delta Q'^{-6}$. Then by (6.1),

$$\sup_{\alpha\in\mathfrak{R}}|H_3^*(\alpha)|\ll P^{1+\varepsilon-\delta/4}R^{5/2}H_sH'M_sM'.$$

But by imitating the proof of [**5**, Lemma 4.10], we readily deduce that

$$\int_0^1|H_3^*(\alpha)|^5d\alpha\ll P^\varepsilon(PH_sH'M_sM')^5{Q'_s}^{-6}.$$

Then on noting that

$$\int_{\mathfrak{R}}|H_3^*(\alpha)|^6d\alpha\ll\sup_{\alpha\in\mathfrak{R}}|H_3^*(\alpha)|\int_0^1|H_3^*(\alpha)|^5d\alpha,$$

the lemma readily follows, in view of our hierarchy Ω.

We now derive an analogue of [**5**, Lemma 12.4].

Lemma 6.3. *Let $H_3(\alpha)$ be defined by* (5.8)*. Then*

$$\int_{\mathfrak{R}} H_3(\alpha)|h_\pi(\alpha,\theta)|^{20}d\alpha \ll P^{1+\varepsilon-\delta/24}M_sM'H_sH'{Q'_s}^{14}.$$

Proof. Suppose that $\alpha\in\mathfrak{R}$. By Dirichlet's theorem we may choose $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ so that $(a,q)=1$, $|q\alpha-a|\le PQ'^{-6}R^{-24}$ and $q\le P^{-1}Q'^6R^{24}$. Moreover, since $\alpha\notin\mathfrak{r}$ we have $q\le P$. Then by Lemma 6.1, whenever $\alpha\in\mathfrak{R}$ we have the bound

$$H_3(\alpha)\ll H_3^*(\alpha)+H_sH'M_sM'P^{3/4+\varepsilon}.$$

Consequently,

$$\int_{\mathfrak{R}} H_3(\alpha)|h_\pi(\alpha,\theta)|^{20}d\alpha \ll P^{3/4+\varepsilon}H_sH'M_sM'I_1+I_2,$$

where

$$I_1=\int_0^1 |h_\pi(\alpha,\theta)|^{20}\,d\alpha,$$

and

$$I_2=\int_{\mathfrak{R}} |H_3^*(\alpha)h_\pi(\alpha,\theta)^{20}|d\alpha.$$

As in the treatment of the expression U_1 in the previous section, we have $I_1\ll {Q'_s}^{\lambda_{10}+\varepsilon}$. Furthermore, by Hölder's inequality, $I_2\ll J_1^{5/6}J_2^{1/6}$, where

$$J_1=\int_0^1 |h_\pi(\alpha,\theta)|^{24}d\alpha,\quad\text{and}\quad J_2=\int_{\mathfrak{R}}|H_3^*(\alpha)|^6d\alpha.$$

By considering the underlying diophantine equations, from (1.1) we have $J_1\ll {Q'_s}^{18+\varepsilon}$, and by Lemma 6.2 we have $J_2\ll P^{\varepsilon-\delta/4}(PH_sH'M_sM')^6{Q'_s}^{-6}$. The lemma now follows with a little calculation.

It remains to deal with the pruned minor arcs $\mathfrak{r}$.

Lemma 6.4. *Let $H_3(\alpha)$ be defined by* (5.8)*. Then*

$$\int_{\mathfrak{r}} H_3(\alpha)|h_\pi(\alpha,\theta)|^{20}d\alpha \ll P^{1-\delta+\varepsilon}H_sH'M_sM'{Q'_s}^{14}.$$

Proof. First observe that on applying [**4**, Lemma 4.1] to (5.8), we obtain

$$H_3(\alpha)\ll P^\varepsilon H_3^+(\alpha),$$

where

$$H_3^+(\alpha)=\sum_{m,u}\sum_{h,g}\sup_{\beta\in[0,1]}\left|\sum_{1\le y\le 4P}e(2^{-6}\alpha\Psi_2(y;2h,g;m,2u)+\beta y)\right|.$$

By standard Weyl differencing we have

$$|H_3^+(\alpha)|^4\ll P^3(H_sH'M_sM'R)^4+P(H_sH'M_sM'R)^3|F_4^+(\alpha)|,$$

where

$$F_4^+(\alpha)=\sum_{m,u}\sum_{h,g}\sum_{1\le h_1\le 4P}\sum_{1\le h_2\le 4P}\sum_{0<z\le 4P-h_1-h_2}e\left(\alpha\Phi_4\right),$$

and

$$\Phi_4=2^{-12}\Psi_4(2z+h_1+h_2,4h,2g,h_1,h_2,m,2u,1,1).$$

Here we have extended the notation introduced in (4.2), so that

$$\Psi_3(z,\mathbf{h},\mathbf{m})=m_3^{-6}\sum_{i=0,1}(-1)^i\Psi_2(z+(-1)^ih_3m_3^6,h_1,h_2,m_1,m_2),$$

and

$$\Psi_4(z,\mathbf{h},\mathbf{m}) = m_4^{-6}\sum_{i=0,1}(-1)^i\Psi_3(z+(-1)^i h_4 m_4^6, h_1,h_2,h_3,m_1,m_2,m_3).$$

We may therefore apply the analysis subsisting in [**5**, Lemma 12.4] (see also the proof of [**5**, Lemma 12.3]), to deduce that

$$\int_{\mathfrak{r}} H_3(\alpha)|h_\pi(\alpha,\theta)|^{20}d\alpha \ll P^{1+\varepsilon}M_sM'H_sH'Z^{-1/8}{Q'_s}^{\lambda_{10}},$$

where

$$Z = PM_s^{27/28}\left(P^{\frac{1}{3}}{M'}^{44-\mu_{28}}\right)^{1/28}.$$

(Note that it is only the minor arc treatment in the latter analysis which we need above, and consequently the secondary term arising in the statement of that lemma may be ignored). The lemma now follows with a little computation.

We are now in a position to provide a suitable estimate for the contribution of the minor arcs $\mathfrak{m}$.

Lemma 6.5. *Let $\mathcal{F}(\alpha)$ be given by* (2.10)*. Then*

$$\sum_{\mathbf{M}}\int_{\mathfrak{m}}|\mathcal{F}(\alpha)|d\alpha \ll P^{18-\varepsilon}.$$

Proof. By Lemmata 6.3 and 6.4, we have

$$U_2(\theta) = \int_{\mathfrak{q}} H_3(\alpha)|h_\pi(\alpha,\theta)|^{20}d\alpha \ll P^{1-\nu}H_sH'M_sM'{Q'_s}^{14}.$$

Then by Lemma 5.2, in view of our hierarchy Ω,

$$\begin{aligned}K_1 &\ll P^{18-\nu}M_s^{-24}{M'}^{-19} + P^{1-\nu+\varepsilon}H_s^2M_s^2H'M'RD^7{Q'_s}^{14}\\ &\ll P^{18-\nu}M_s^{-24}{M'}^{-19}.\end{aligned}$$

Next, by combining the conclusions of Lemmata 4.4, 4.5 and 4.6, we deduce that

$$\int_{\mathfrak{p}}|\mathcal{H}(\alpha)g_s(\alpha)^{21}|d\alpha \ll P^{18-\nu}M_s^{-21}.$$

Thus, by Lemmata 3.3, 4.2 and 4.3, we have

$$M_s^{21}\mathcal{J}_s(\mathfrak{n}) \ll P^{18-\varepsilon},$$

and consequently the lemma follows from Lemma 3.1, and the observation that $\sum_{\mathbf{M}}1 \ll (\log P)^{22}$.

7. The major arcs.

Let $\mathfrak{M}$ denote the major arcs $(Q^{-6(1-\sigma)}, 1+Q^{-6(1-\sigma)}]\setminus\mathfrak{m}$, that is, the union of the intervals

$$\mathfrak{M}(q,a) = \{\alpha\in(Q^{-6(1-\sigma)},1+Q^{-6(1-\sigma)}]\ :\ |\alpha q - a|\le Q^{-6(1-\sigma)}\}$$

with $1\le a\le q\le Q^{6\sigma}M^6$ and $(a,q)=1$. The proof of the theorem will be completed if we can show that

$$\sum_{\mathbf{M}}\int_{\mathfrak{M}}\mathcal{F}(\alpha)e(-\alpha n)d\alpha \gg P^{18}.$$

Let W denote a parameter to be chosen later, and let $\mathfrak{N}$ denote the union of the intervals

$$\mathfrak{N}(q,a) = \left\{\alpha\in(Q^{-6(1-\sigma)},1+Q^{-6(1-\sigma)}]\ :\ |\alpha q-a|\le WP^{-\delta}\right\},$$

with $(a,q)=1$ and $1\le a\le q\le W$. We assume that $1\le W\le P^{1/2}$, so that $\mathfrak{N}\subset\mathfrak{M}$, and take $\mathfrak{Q}=\mathfrak{M}\setminus\mathfrak{N}$. We define $V(\alpha)$ on $\mathfrak{M}$ by taking

$$V(\alpha) = q^{-1}S(q,a)v(\alpha-a/q)\quad(\alpha\in\mathfrak{M}(q,a))$$

where

$$v(\beta) = \sum_{1\le x\le n}\tfrac{1}{6}x^{-5/6}e(\beta x)$$

and

$$S(q,a) = \sum_{r=1}^{q}e\left(ar^6/q\right).$$

Lemma 7.1. *We have*

$$\sum_{\mathbf{M}} r(n;\mathbf{M}) = \int_{\mathfrak{M}} V(\alpha)^2 \left(\prod_{s=1}^{22} \sum_{M_s} \sum_{p_s} g_s\left(\alpha p_s^6\right)\right) e(-\alpha n) d\alpha + O\left(P^{18-\varepsilon}\right).$$

Proof. Write $\Delta(\alpha) = |F(\alpha) - V(\alpha)|$. Then by [**1**, Theorem 2], we have

$$\Delta(\alpha) \ll q^{\varepsilon}\left(q + P^6|\alpha q - a|\right)^{1/2}$$

whenever $\alpha \in \mathfrak{M}(q,a)$. Hence, for $\alpha \in \mathfrak{M}$ we have $\Delta(\alpha) \ll Q^{3\sigma+\varepsilon}M^3$. Then

$$\int_{\mathfrak{M}} \Delta(\alpha)^2 \left|\sum_{p_s} g_s\left(\alpha p_s^6\right)\right|^{22} d\alpha \ll Q^{6\sigma+\varepsilon}M^6\mathcal{I}_{1,s},$$

and by Lemma 2.1, a little calculation establishes the latter to be $\ll P^{18-\delta}$. Also, by Vaughan [**2**, Lemma 4.6], we have

$$V(\alpha) \ll P(q + P^6|\alpha q - a|)^{-1/6} \quad (\alpha \in \mathfrak{M}(q,a)),$$

and hence

$$V(\alpha)\Delta(\alpha) \ll P^{1+\varepsilon}\left(Q^{6\sigma}M^6\right)^{1/3}.$$

Thus, with a little calculation, once again we deduce that

$$\int_{\mathfrak{M}} \left|V(\alpha)\Delta(\alpha)\left(\sum_{p_s} g_s\left(\alpha p_s^6\right)\right)^{22}\right| d\alpha \ll P^{18-\delta}.$$

The lemma now follows immediately.

We now consider

$$K = \int_0^1 \left|\sum_{M_s}\sum_{p_s} g_s\left(\alpha p_s^6\right)\right|^{26} d\alpha.$$

By examining the underlying diophantine equation we see that

$$K \le \int_0^1 |H(\alpha;2P)|^4 \left|\sum_{M_s}\sum_{p_s} g_s\left(\alpha p_s^6\right)\right|^{22} d\alpha.$$

Thus by combining Lemma 2.1 with a standard application of the Hardy-Littlewood method (see [**5**, Lemma 7.3]) we obtain $K \ll P^{20}$. It now follows by Hölder's inequality and standard estimates for $\int_{\mathfrak{Q}} |V(\alpha)|^{13} d\alpha$ that

$$\int_{\mathfrak{Q}} \left|V(\alpha)^2 \prod_{s=1}^{22}\left(\sum_{M_s}\sum_{p_s} g_s\left(\alpha p_s^6\right)\right)\right| d\alpha \ll P^{18}W^{-\nu}.$$

By the methods of [**3**, section 5], when $W \le \log P$, $q \le \log P$, and $(a,q) = 1$, we have

$$\sum_{p_s} g_s(\alpha p_s^6) = q^{-1}S(q,a)u_s(\alpha - a/q) + O\left(\frac{P}{\log P}(q + P^6|\alpha q - a|)\right)$$

where

$$u_s(\beta) = \sum_{x \le (2P)^6} \frac{\min\{\log(2Px^{-1/6}), \log 2\}}{2\log M_s} \tfrac{1}{6}x^{-5/6}\rho\left(\frac{\log(x^{1/6}/M_s)}{\log R}\right) e(\beta x),$$

and $\rho(x)$ is Dickman's function, defined for real x by

$$\rho(x) = 0 \text{ when } x \le 0,$$

$$\rho(x) = 1 \text{ when } 0 < x \leq 1,$$

$$\rho \text{ is continuous for } x > 0,$$

$$\rho \text{ is differentiable for } x > 1,$$

$$x\rho'(x) = -\rho(x-1) \text{ for } x > 1.$$

In particular, when $x \geq 0$, $\rho(x)$ is positive and decreasing. Thus if we take ϕ sufficiently small, and $W = (\log P)^\phi$, then in the usual way we obtain

$$\int_{\mathfrak{N}} V(\alpha)^2 \left(\prod_{s=1}^{22} \sum_{p_s} g_s(\alpha p_s^6) \right) e(-\alpha n) d\alpha = \mathfrak{S}(n) J(n) + O\left(P^{18} (\log P)^{-22-\delta}\right),$$

where $\mathfrak{S}(n)$ is the usual singular series in Waring's problem,

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left(q^{-1} S(q,a)\right)^{24} e(-an/q),$$

and

$$J(n) = \int_0^1 v(\beta)^2 \left(\prod_{s=1}^{22} u_s(\beta) \right) e(-\beta n) d\beta.$$

Now by [**2**, Theorem 4.6], we have $1 \ll \mathfrak{S} \ll 1$, and moreover a simple counting argument shows that $J(n) \gg n^3 (\log n)^{-22}$. Thus

$$\begin{aligned} \sum_{\mathbf{M}} r(n; \mathbf{M}) &= \sum_{\mathbf{M}} \int_{\mathfrak{N}} V(\alpha)^2 \left(\prod_{s=1}^{22} \sum_{p_s} g_s(\alpha p_s^6) \right) e(-\alpha n) d\alpha + O\left(P^{18} (\log P)^{-\delta}\right) \\ &\gg n^3, \end{aligned}$$

and this completes the proof of our theorem.

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